A NOTE ON GEOMETRIC AGGREGATION OPERATORS IN T-SPHERICAL FUZZY ENVIRONMENT AND THEIR APPLICATIONS IN MULTI-ATTRIBUTE DECISION MAKING

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ABSTRACT

Describing uncertainties of more than one aspect is a hot research topic in fuzzy mathematics. Atanassov's intuitionistic fuzzy set (IFS) and Cuong's picture fuzzy set (PFS) are two featured fuzzy concepts. Recently, a novel framework of T-spherical fuzzy set (TSFS) and consequently spherical fuzzy set (SFS) are developed for handling those problems where uncertain situations have more than two aspects. This manuscript is based on some contribution to the area of SFS and TSFS. In this manuscript, some properties of aggregation tools of TSFS (SFS) are discussed and some ordered weighted geometric (OWG) and hybrid geometric (HG) operators are developed. It is discussed that these aggregation operators are generalizations of the aggregation operators of IFSs and PFSs. Multi-attribute decision making (MADM) process is comprehensively discussed in T-spherical fuzzy environment and elaborated with a numerical example. The results obtained are analyzed and their advantages over existing structures are studied.

KEYWORDS: *Picture fuzzy sets. T-spherical fuzzy sets. Aggregation operators. Decision making.*

INTRODUCTION

Since the discovery of Zadeh's FS theory (Zadeh, 1965), many real-life problems having uncertainty have been modeled using FS theory. Among famous applications of FS theory decision making (DM) is one of them in which different extensions of FS theory have been applied quite successfully. After the commencement of FS theory some quality work in the area of DM have been done such as (Chen and Tan, 1994) deals with handling multi-criteria fuzzy DM problems, (Bellman and Zadeh, 1970) uses DM techniques in management sciences using FS theory, (Zimmermann, 2012; Kacprzyk and Fedrizzi, 2012) also discussed some solid aspects of DM problems. In a DM problem our main goal is to aggregate the data provided in a situation and to do so one need aggregation tools. To aggregate the data in fuzzy environment some quality of work is being done by various researchers (Zimmermann, and Zysno, 1980; Grabisch, 1995; Yager, 1996; Calvo *et al.,* 2012; Mahmood *et al.,* 2018; Mahmood *et al.,* 2017).

Zadeh's frame work of FS is a generalization of crisp sets and deals with uncertain events by associating the membership grades of an element with a number from closed interval on a scale of to . In FSs, one can only assign values to show the extent of belongingness from a scale of to while its degree of non-membership can be found by default by subtracting the membership degree form 1 sec . Feeling the need of a structure where one can not only describe the membership factor but also describe its non-membership independently, Atanassov initiated the theory of IFSs (Atanassov, 1986). This new structure of Atanassov give the access to discuss the membership and non-membership of an element with a constraint that their sum must belongs to 0 (Zadeh, 1965). The concept of IFS is of great significance and it has been used in DM problems successfully as (Xu, 2007; Xu, and Yager, 2006; Wei, 2010) defined aggregation operator for IFSs, (Zhao *et al.,* 2010) discussed generalized aggregation tools for IFSs and apply them in DM problems. For other work in this direction, one may refer to (Beliakov *et al.,* 2011; Li, 2010; Wei, G. and Wang. 2007; Xu, and Xia, 2011). The constraint on IFSs that the sum of membership and non-membership must lies in somehow bound us to remain in a certain domain so one is unable to assign grades of membership and non-membership but from a certain domain. Keeping in view this factor in 2013 (Yager, 2013) proposed the concept of Pythagorean FSs which mainly enlarge the domain of IFS. For a detailed study in these directions, one may refer to (Yager, 2013; Yager, 2014; Peng, and Yang, 2015; Garg, 2016; Ullah *et al.,* 2018; Jan *et al.,* 2018; Davvaz *et al.,* 2018; Khan *et al.,* 2018; Jan *et al.,* 2018).

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There is another shortcoming of Atanassov's IFSs that is it described only affiliation and disaffiliation degree. In the phenomena of voting, it is observed that voting terminology has more than two aspects which are vote (in favor or abstain or against or refusal). So ordinary IFSs is unable to model such type of events. Meeting the need, B. C. Cuong (Cuong, 2014) proposed the idea of PFSs which is based on four components describing membership, abstinence, non-membership and refusal degree of an element. Such type of model can best describe real life phenomena due to its diverse structure. PFS proved to be a useful tool as recently a lot of work is being done in this area. Some of the recent developments of PFS are in (Thong, 2014; Singh, 2015; Thong, 2015; Wei, 2016; Garg, 2017; Wei, 2017; Wang *et al.,* 2017). Although the concept of PFS can model human behavior better than the existing ideas but still there is a problem with its constraint that the sum of membership, abstinence and non-membership grades should belongs to . Due to this problem the decision makers could not assign values by their own consent and somehow, they are bound in a certain domain. Keeping this point in mind Mahmood et al (Mahmood *et al.,* 2018) proposed a novel concept of TSFSs (SFSs) which enlarged the domain of PFS and enables the decision makers for assigning values of their own choice freely. The frameworks of TSFSs (SFSs) are discussed in section 2 briefly. To enable the framework of TSFSs, Mahmood et al. (2018) also developed the geometric aggregation tools for TSFSs and applied those tools in MADM. TSFSs are further utilized in (Ullah *et al.,* 2018) where some similarity measures are proposed for TSFSs and applied in building material recognition problem. In (Garg *et al.,* 2018), the geometric interactive aggregation operators of TSFSs are developed and MADM problem is investigated using proposed operators. The idea of weighted averaging aggregation operators for TSFSs are developed by Ullah et al. (2019) which are further utilized in MADM problems.

In this manuscript, we first improved the geometric aggregation operators studied by Mahmood et al. (2018). We also studied some OWG operators and HG operators for TSFSs (SFSs) and elaborated with examples. The significance of proposed operators over the existing operators are discussed and it is shown that the developed operators are the generalization of the pre-existing aggregation tools of PFSs as well as IFSs and the result obtained here are better than obtained previously. A detailed analysis of these results is also included in the manuscript.

This paper consists of 7 sections where section one is based on history of various fuzzy frameworks and their developments. The novelty of TSFSs over other fuzzy frameworks is also studied. In section two, basic concepts are explained along with the description of spaces of TSFS (SFS) geometrically. Section three is based on some analysis of aggregation operators developed in (Mahmood *et al.,* 2018) for TSFSs (SFSs) along with their properties. Section four is based on OWG and HG operators for TSFSs. In section five, DM process is discussed and explained with the help of numerical example. Some discussion about the results of proposed work and its advantages are included in section six. In section seven, the article is summarized, and some future directions are studied.

Preliminaries

The concept of IFSs, PFSs, SFSs and T-SFSs are discussed in this section along with some of their operations. Some aggregation operators of T-SFSs are also discussed whose properties are discussed in next section.

Definition 1: (Mahmood et al., 2018) A TSFS on a set X is having the shape $T=\{x,s(x),i(x),d(x)\, for\, all\, x\in X\}$ where $s,i,d{:}X\to[0,1]$ represent the membership, abstinence and non-membership grade respectively such that for some $0\leq S^n(x)+i^n(x)+d^n(x)\leq 1$. The term $r(x)=\sqrt[n]{1-\left(s^n(x)+i^n(x)+d^n(x)\right)}$ is known as the refusal degree of x in S. The triplet (s,i,d) is known as T-spherical fuzzy number (TSFN).

Remark 1: The Definition 1 becomes the definition of:

1. SFS if n is taken as 2.
2. PFS if n is taken as 1.
3. PyFS if n is taken as 2 and $i = 0$.
4. IFS if n is taken as 1 and $i = 0$.

Clearly the Definition 1 and Remark 1 gave us the information about the spaces of IFSs, PFSs, SFSs and TSFSs. A graphical comparison of their spaces was

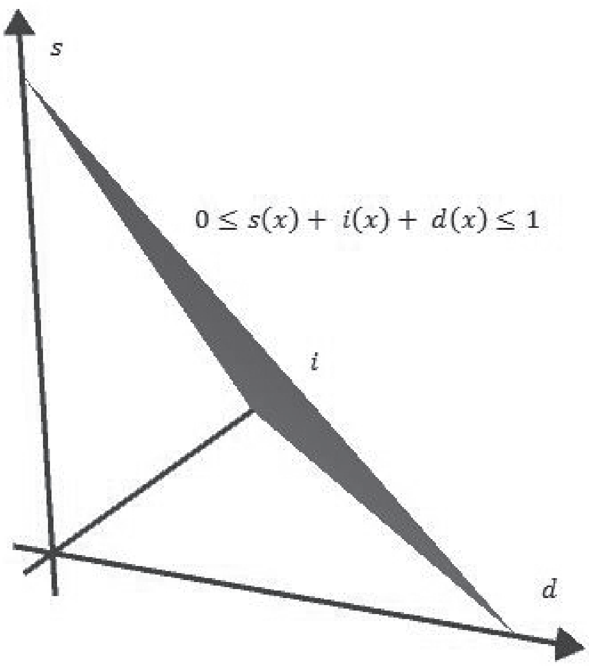

Fig 1: (Space of picture fuzzy membership grades)

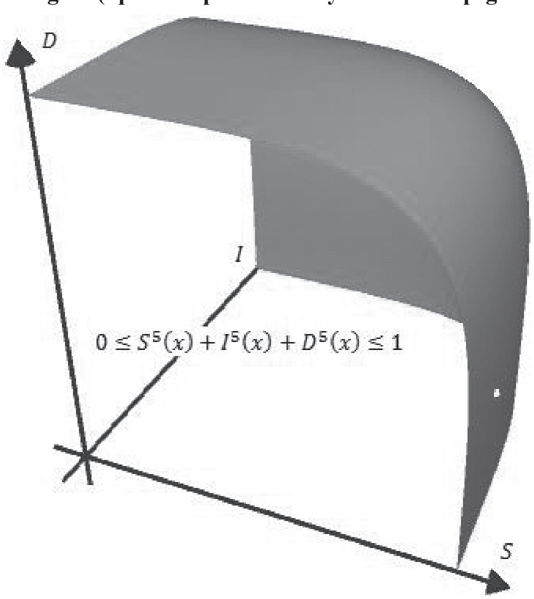

Fig 2: (space of spherical fuzzy membership grades)

portrayed in (Wei, 2016) which is improved and new geometrical representation is presented below for further clarification these concepts.

All these figures indicated that the framework of T-SFSs is an improved fuzzy model than that of IFSs, PFSs, SFSs and has no limitation. Such a framework could be very useful in handling real-life problems.

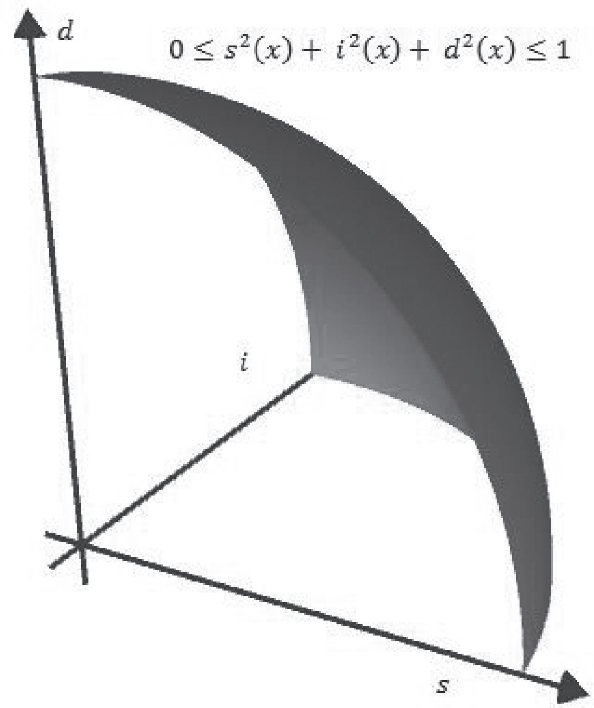

Fig 3: (T-SFS for n=5)

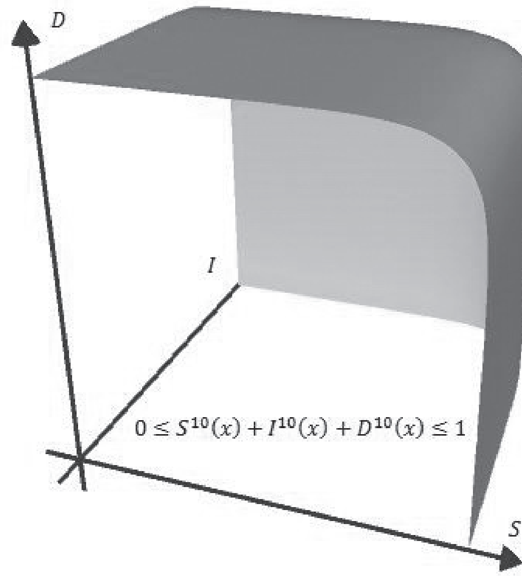

Fig 4: (T-SFS for n=10)

Definition 5: (Mahmood *et al.,* 2018) For two TSFNs $T_A = (s_A, i_A, d_A)$ and TB = (s_B,i_B,d_B) and for $\lambda > 0$

$$T_A.T_B = \left((s_A + i_A)(s_B + i_B) - i_A.i_B, i_A.i_B, \sqrt[n]{1 - \left(1 - d_A^n\right)\left(1 - d_B^n\right)} \right)$$

$$T_A^\lambda = \left(\left(s_A + i_A\right)^\lambda - i_A^\lambda, i_A^\lambda, \sqrt[n]{1 - \left(1 - d_A^n\right)^\lambda} \right)$$

$If\, T_A = (s_A, i_A, d_A)\, and T_B = (s_B, i_B, d_B)$ be two SFNs. Then

$$T_A.T_B = \left((s_A + i_A)(s_B + i_B) - i_A.i_B, i_A.i_B, \sqrt{1-(1-d_A^2)(1-d_B^2)}\right)$$

$$T_A^\lambda = \left((s_A + i_A)^\lambda - i_A^\lambda, i_A^\lambda, \sqrt{1-(1-d_A^2)^\lambda}\right)$$

The operations defined in Definition 5 are direct generalization of the picture fuzzy operation proposed in 2017 (Wang *et al.,* 2017). Further, taking $i_A = i_B = 0$ reduces the operation of Definition 5 to the environment of IFSs.

In Definition 6, the concept of score value of a TSFNs is elaborated followed by their ranking principle. Using some suitable restrictions, this definition of score function can easily be transformed to the environment of PFSs as discussed in (Wang *et al.,* 2017). The accuracy function for comparison of two or more TSFN is also discussed which can be used in a case where score function could not differentiate between two TSFNs.

Definition 6: (Mahmood *et al.,* 2018) The score of a TSFN T is defined as SC (T) = S^n (x) - d^n (x) and SC (A) $\in$ [-1,1]. Based on this rule, for two TSFNs T_1 and T_2:

- T_1 is superior to $T_2\, if\, SC(T_1) > SC(T_2)$.
- T_1 is inferior to $T_2\, if SC(T_1) < SC(T_2)$.

If SC(T_1) = SC(B) for two SFNs. Then, we need to differentiate between them using accuracy function which is defined as AC(A) $= s_A^n(x) + i_A^n(x) + d_A^n(x)$ and $AC(A) \in [0,1]$ For two TSFNs T_1 and T_2:

- T_1 is superior to T_2 if $C(T_1) > AC(T_2)$.
- T_1 is inferior to T_2 if $AC(T_1) < AC(T_2)$.
- A is similar to B if $AC(A) = AC(B)$.

Definition 7: (Mahmood *et al.,* 2018) The geometric aggregation operators for a number of TSFNs TJ = (j=1,2,3,...m) is denoted by TSFWG and is defined as:

$$T - SFWG(T_1, T_2, \ldots T_m) = \prod_{j=1}^{m} T_j^{w_j}$$

where w is the weight vector such that w_j > 0 and $\sum_{j=1}^{m} w_j = 1$.

A more comprehensive form of the geometric aggregation operator can be constructed using Definition 5 given in Theorem 1.

Theorem 1: (Mahmood et al., 2018) A collection of TSFNs Tj (j=1,2,3,...m), if aggregated using TSFWG operator, gives a TSFN as:

$$T - SFWG(T_1, T_2, \ldots T_m) = \left(\prod_{j=1}^{m}(s_j^n + i_j^n)^{w_j} - \prod_{j=1}^{m} i_j^{n w_j}, \prod_{j=1}^{m} i_j^{n w_j}, \sqrt[n]{1-\prod_{j=1}^{m}(1-d_j^n)^{w_j}}\right)$$

The operation defined in Theorem 1 is derived based on Definition 5. We believe that there is a mistake in defining the operation, the way it is defined in Theorem 1. In this manuscript, we proposed a new version of this geometric operation which is indeed based on Definition 5.

Analysis of Geometric Aggregation Operators for T-Spherical Fuzzy Sets:

In this section, first we defined the geometric operation in a new way. Then we studied some properties of geometric operators like idempotency, boundedness and monotonicity etc. The fitness of the new proposed operation is also discussed with the help of mathematical induction. The following Theorem 3 is the new improved form of geometric operators defined in (Wei, G. 2016).

Theorem 2: For TSFNs Tj(j=1,2,3,...m), the aggregated value by applying TSFWG operator is a TSFN and

$$T - SFWG(T_1, T_2, \ldots T_m) = \left(\prod_{j=1}^{m}(s_j + i_j)^{w_j} - \prod_{j=1}^{m} i_j^{w_j}, \prod_{j=1}^{m} i_j^{w_j}, \sqrt[n]{1-\prod_{j=1}^{m}(1-d_j^n)^{w_j}}\right)$$

Theorem 3: (Idempotency) If in a collection of TSFNs $T_j\,(j = 1,2,3,\ldots m), T_1 = T_2 = T_3 = \ldots = T_m = T$. Then

$$T - SFWG(T_1, T_2, \ldots T_n) = T$$

Proof: Let $T_1 = T_2 = T_3 = \ldots = T_m = T = (s, i, d)$. Then by theorem 1, we have

$$T - SFWG(T_1, T_2, \ldots T_m) = \left(\prod_{j=1}^{m}(s_j + i_j)^{w_j} - \prod_{j=1}^{m} i_j^{w_j}, \prod_{j=1}^{m} i_j^{w_j}, \sqrt[n]{1-\prod_{j=1}^{m}(1-d_j^n)^{w_j}}\right)$$

As $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Hence

$$T - SFWG(T, T, \ldots T) = ((s+i) - i, i, 1-(1-v)) = (s, i, d) = T$$

Theorem 4: (Boundedness) Consider a collection of

TSFNs T_j(j=1,2,3,...m) such that $T^U=\left(s_j^U,i_j^U,d_j^U,r_j^U\right)$ denote the maximum value and $T^L=\left(s_j^L,i_j^L,d_j^L,r_j^L\right)$ denote the minimum value $\forall j$. Then

$$T^L \le T-SFWG\left(T_1,T_2,\ldots T_m\right)\le T^U$$

Proof:

As

$$T-SFWG\left(T_1,T_2,\ldots T_m\right)=\left(\prod_{j=1}^{m}\left(s_j+i_j\right)^{w_j}-\prod_{j=1}^{m}i_j^{w_j},\prod_{j=1}^{m}i_j^{w_j},\sqrt[n]{1-\prod_{j=1}^{m}\left(1-d_j^n\right)^{w_j}}\right)$$

Consider $\left(s_j+i_j\right)^{w_j}=\left(1-d_j-r_j\right)^{w_j}\le\left(1-d^L-r^L\right)^{w_j}$

$\left(s_j+i_j\right)^{w_j}=\left(1-d_j-r_j\right)^{w_j}\ge\left(1-d^U-r^U\right)^{w_j}$

And $i^L\le i\le i^U$

Hence

$$\prod_{j=1}^{m}\left(1-d^L-r^L\right)^{w_j}-\prod_{j=1}^{m}\left(i_j^L\right)^{w_j}$$
$$\le\prod_{j=1}^{m}\left(s_j+i_j\right)^{w_j}-\prod_{j=1}^{m}i_j^{w_j}$$
$$\le\prod_{j=1}^{m}\left(1-d^U-r^U\right)^{w_j}-\prod_{j=1}^{m}\left(i_j^U\right)^{w_j}$$

As $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Hence

$$s^L=\left(1-d^L-r^L-i^L\right)$$
$$\le\prod_{j=1}^{m}\left(s_j+i_j\right)^{w_j}-\prod_{j=1}^{m}i_j^{w_j}$$
$$\le\left(1-d^U-r^U\right)-i_j^U=s^U$$

This implies that

$$s^L\le s\le s^U$$

Hence

$$s^L\le s\le s^U$$

Also

$$i^L\le i\le i^U$$

Similarly,

$$d^L\le\sqrt[n]{1-\prod_{j=1}^{m}\left(1-d_j^n\right)^{w_j}}\le d^U$$

This implies that

$$d^L\le d\le d^U$$

Which means that

$$T^L\le T-SFWG\left(T_1,T_2,\ldots T_m\right)\le T^U$$

Theorem 5: (Monotonicity) Consider two collections of TSFNs $T_j=\left(s_j,i_j,d_j,r_j\right)$ and $T_j^{'}=\left(s_j^{'},i_j^{'},d_j^{'},r_j^{'}\right)(j=1,2,3,\ldots m)$ such that for $1\le j\le m i_j\le i_j^{'},d_j\le d_j^{'}$ and $r_j\le r_j^{'}$. Then $T-SFWG\left(T_1,T_2,\ldots T_m\right)\ge T-SFWG\left(T_1^{'},T_2^{'},\ldots T_m^{'}\right)$

Proof: Using provided information

$$\prod_{j=1}^{m}\left(s_j+i_j\right)^{w_j}-\prod_{j=1}^{m}\left(i_j\right)^{w_j}$$
$$=\prod_{j=1}^{m}\left(1-d_j-r_j\right)^{w_j}-\prod_{j=1}^{m}\left(i_j\right)^{w_j}$$
$$\ge\prod_{j=1}^{m}\left(1-d_j^{'}-r_j^{'}\right)^{w_j}-\prod_{j=1}^{m}\left(r_j^{'}\right)^{w_j}$$
$$=\prod_{j=1}^{m}\left(s_j^{'}+i_j^{'}\right)^{w_j}-\prod_{j=1}^{m}\left(r_j^{'}\right)^{w_j}$$

Further

$$1-\prod_{j=1}^{m}\left(1-d_j\right)^{w_j}\le1-\prod_{j=1}^{m}\left(1-d_j^{'}\right)^{w_j}$$

Hence

$$T-SFWG\left(T_1,T_2,\ldots T_m\right)\ge T-SFWG\left(T_1^{'},T_2^{'},\ldots T_m^{'}\right)$$

Which proves monotonicity.

Ordered Weighted and Hybrid Geometric Operators for T-Spherical Fuzzy Sets:

In aggregation, sometimes we need to weight the ordered position of argument instead of weighting the argument. For those situations, here in this section some OWG operators are defined. Further, when both the argument and its ordered position are required to be weighted we introduced the concept of HG operators.

Definition 8: For a collection of TSFNs $T_j\left(j=1,2,3,\ldots m\right)$ the T-SFOWG operator is defined as:

$$T-SFOWG(T_1,T_2,\ldots T_m)=\prod_{j=1}^{m}\left(T_{\Omega(j)}\right)^{w_j}$$

Here $w=(w_1,w_2,\ldots w_m)^T$ be a weight vector and $w_j>0$ *and* $\sum_{j=1}^{m}w_j=1$ and $T_{\Omega(j)}$ is the *j-th* largest element of T-SF arguments.

By applying the operational laws of T-SFSs, the following theorem is helpful in aggregating T-SF information.

Theorem 6:

The aggregated valued of some TSFNs $T_j=(s_j,i_j,d_j,r_j)(j=1,2,,3,\ldots m)$ using *T - SFOWG* operator is a TSFN and

$$T-SFOWG(T_1,T_2,\ldots T_m)=\begin{pmatrix}\prod_{j=1}^{m}\left(s_{\Omega(j)}+i_{\Omega(j)}\right)^{w_j}-\prod_{j=1}^{m}i_{\Omega(j)}{}^{w_j},\\ \prod_{j=1}^{m}i_{\Omega(j)}{}^{w_j},\ \sqrt[n]{1-\prod_{j=1}^{m}\left(1-d_{\Omega(j)}^{n}\right)^{w_j}}\end{pmatrix}$$

Proof: For m=2

$$T_{\Omega(1)}^{w_1}=\left(\left(s_{\Omega(1)}+i_{\Omega(1)}\right)^{w_1}-i_{\Omega(1)}^{w1},i_{\Omega(1)}^{w1},\sqrt[n]{1-\left(1-d_{\Omega(1)}^{n}\right)^{w_1}}\right)$$

$$T_{\Omega(2)}^{w_2}=T_{\Omega(2)}^{w_2}=\left(\left(s_{\Omega(2)}+i_{\Omega(2)}\right)^{w_2}-i_{\Omega(1)}^{w2},i_{\Omega(1)}^{w2},\sqrt[n]{1-\left(1-d_{\Omega(2)}^{n}\right)^{w_2}}\right)$$

Then

$$T_{\Omega(1)}^{w_1}.T_{\Omega(2)}^{w_2}=\begin{pmatrix}\left(\left(s_{\Omega(1)}+i_{\Omega(1)}\right)^{w_1}-i_{\Omega(1)}^{w1}\right)\\ \left(\left(s_{\Omega(2)}+i_{\Omega(2)}\right)^{w_2}-i_{\Omega(1)}^{w2}\right)-i_{\Omega(1)}^{w1}.i_{\Omega(2)}^{w2},\\ i_{\Omega(1)}^{w1}.i_{\Omega(2)}^{w2},\sqrt[n]{1-\left(1-d_{\Omega(1)}^{n}\right)^{w_1}.\left(1-d_{\Omega(2)}^{n}\right)^{w_2}}\end{pmatrix}$$

Thus, result is true when *m=2*. Assume that result is true for *m=k* i.e.

$$TSFOWG(T_1,T_2,\ldots T_k)=$$
$$\left(\prod_{j=1}^{k}\left(s_{\Omega(j)}+i_{\Omega(j)}\right)^{w_j}-\prod_{j=1}^{k}i_{\Omega(j)}{}^{w_j},\prod_{j=1}^{k}i_{\Omega(j)}{}^{w_j},\sqrt[n]{1-\prod_{j=1}^{k}\left(1-d_{\Omega(j)}^{n}\right)^{w_j}}\right)$$

To prove the result for m = k+1

$$TSFOWG(T_1,T_2,\ldots T_{k+1})=\prod_{j=1}^{k+1}T_j^{w_j}$$

$$=\prod_{j=1}^{k}T_j^{w_j}.T_{k+1}^{w_{k+1}}$$

$$=\left(\prod_{j=1}^{k}\left(s_{\Omega(j)}+i_{\Omega(j)}\right)^{w_j}-\prod_{j=1}^{k}i_{\Omega(j)}{}^{w_j},\prod_{j=1}^{k}i_{\Omega(j)}{}^{w_j},\ \sqrt[n]{1-\prod_{j=1}^{k}\left(1-d_{\Omega(j)}^{n}\right)^{w_j}}\right).$$
$$\left(\left(s_{k+1}+i_{k+1}\right)^{w_{k+1}}-i_{k+1}^{w_{k+1}},i_{k+1}^{w_{k+1}},\ \sqrt[n]{1-\left(1-d_{k+1}^{n}\right)^{w_{k+1}}}\right)$$

$$TSFOWG(T_1,T_2,\ldots T_{k+1})$$
$$=\left(\prod_{j=1}^{k+1}\left(s_{\Omega(j)}+i_{\Omega(j)}\right)^{w_j}-\prod_{j=1}^{k+1}i_{\Omega(j)}{}^{w_j},\prod_{j=1}^{k+1}i_{\Omega(j)}{}^{w_j},\sqrt[n]{1-\prod_{j=1}^{k+1}\left(1-d_{\Omega(j)}^{n}\right)^{w_j}}\right)$$

Remark 2: Theorem 6 becomes valid in spherical fuzzy environment if is replaced by .

The following results holds for T-SFOWG operators. The proofs are omitted as these proofs are already discussed in section 3.

Theorem 7: (Idempotency) If in a collection of TSFNs $T_j\ (j=1,2,3,\ldots m),T_1=T_2=T_3=\ldots=T_m=T$. Then

$$T-SFOWG(T_1,T_2,\ldots T_n)=T$$

Theorem 8: (Boundedness) Consider a collection of TSFNs $T_j\ (j=1,2,3,\ldots m)$ such that $T^U=(s_j^U,i_j^U,d_j^U,r_j^U)$ denote the maximum value and $T^L=(s_j^L,i_j^L,d_j^L,r_j^L)$ denote the minimum value $\forall j$. Then

$$T^L\leq T-SFOWG(T_1,T_2,\ldots T_m)\leq T^U$$

Theorem 9: (Monotonicity) Consider a collection of TSFNs $T_j=(s_j,i_j,d_j,r_j)$ and $\dot{T}_j=\left(\dot{s}_j,\dot{i}_j,\dot{d}_j,\dot{r}_j\right)(j=1,2,3,\ldots m)$ such that $T_j\subseteq \dot{T}_j$. Then

$$T-SFOWG(T_1,T_2,\ldots T_m)\leq T-SFOWG\left(\dot{T}_1,\dot{T}_2,\ldots \dot{T}_m\right)$$

Ordered weighted averaging operators for T-SFSs.

Definition 9: For a collection of TSFNs $T_j\ (j=1,2,3,\ldots m)$ the T-SFHG operator is defined as:

$$T-SFHG(T_1,T_2,\ldots T_m)=\prod_{j=1}^{m}\left(\ddot{T}_{\Omega(j)}\right)^{w_j}$$

Here $w=(w_1,w_2,\ldots w_m)^T$ be the aggregation associated weight vector $w_j>0$ and $\sum_{j=1}^{m}w_j=1$ and $T_{\Omega(j)}$ and is the j-th largest element of T-SF arguments. Also $\ddot{T}_j=\left(\ddot{T}_j\right)^{l\dot{w}_j}$ where $\dot{w}=(\dot{w}_1,\dot{w}_2,\ldots \dot{w}_m)^T$ be the weight vector of the T-SF arguments and l is the balancing factor.

From the operational laws of T-SFSs, we have

Theorem 10:

The aggregated valued of some TSFNs $T_j=(s_j,i_j,d_j,r_j)(j=1,2,,3,\ldots m)$ using T−SFHG operator is a TSFN and

$$T-SFHG\left(T_1,T_2,\ldots T_m\right)$$
$$=\left(\prod_{j=1}^{m}\left(\ddot{s}_{\Omega(j)}+\ddot{i}_{\Omega(j)}\right)^{w_j}-\prod_{j=1}^{m}\ddot{i}_{\Omega(j)}^{\ w_j},\prod_{j=1}^{m}\ddot{i}_{\Omega(j)}^{\ w_j},\sqrt[n]{1-\prod_{j=1}^{m}\left(1-\ddot{d}_{\Omega(j)}^{\ n}\right)^{w_j}}\right)$$

Proof: Proof is follows from Theorem 6.

Remark 3: Replacing n by 2 reduces Theorem 9 for SFSs.

The following results holds for T-SFHG operators. The proofs are omitted as these proofs are already discussed in section 3.

Theorem 11: (Idempotency) If in a collection of TSFNs $T_j\left(j=1,2,3,\ldots m\right), T_1=T_2=T_3=\ldots=T_m=T$. Then

$$T-SFHG\left(T_1,T_2,\ldots T_n\right)=T$$

Theorem 12: (Boundedness) Consider a collection of TSFNs $T_j\left(j=1,2,3,\ldots m\right)$ such that $T^U=\left(s_j^U,i_j^U,d_j^U,r_j^U\right)$ denote the maximum value and $T^L=\left(s_j^L,i_j^L,d_j^L,r_j^L\right)$ denote the minimum value $\forall j$. Then

$$T^L\leq T-SFHG\left(T_1,T_2,\ldots T_m\right)\leq T^U$$

Theorem 13: (Monotonicity) Consider a collection of TSFNs $T_j=\left(s_j,i_j,d_j,r_j\right)$ and $T_j'=\left(s_j',i_j',d_j',r_j'\right)\left(j=1,2,3,\ldots m\right)$ such that $T_j\subseteq T_j'$. Then

$$-SFHG\left(T_1,T_2,\ldots T_m\right)\leq T-SFHG\left(T_1',T_2',\ldots T_m'\right)$$

Multi-Attribute Decision Making

This section is about the description of MADM process. In MADM, generally there are alternatives with j attributes. Let the set of alternatives be $P=\left\{p_j : j=1,2,3\ldots m\right\}$ and the set of attributes be $E=\left\{e_j : j=1,2,3\ldots m\right\}$. The decision makers evaluated the alternatives and provided their evaluations in a decision matrix based on TSFNs. Utilizing the aggregation operators defined in previous section, the information of decision makers is aggregated and by using the ranking function, the selection of appropriate alternative is obtained.

The detailed steps of the algorithm of MADM are explained in a hypothetical numerical example below.

Example 1:

A well-known company is about to launch a new product on the occasion of new year. There are four products that the company have and are ready to be launched. The governing board of the company will decide about which product is to be launched. A member of the governing board will give their opinion about each product in terms of TSFNs i.e. their opinion could be four dimensional as opinion in favor, remain abstain, opinion against or refuse to give opinion. Let $P=\left\{p_i : i=1,2,3,4\right\}$ be the four products and are evaluated under four attributes which are:

$1.Demand.2.PriceofProduct.3.Costumersattraction.$
$4.Marketcompetition.w=(0.35,0.25,0.25,0.15)^T$ be the weight of attributes. The members of governing board gave their opinion while remain anonymously in the form of a decision matrix. All the step of DM process with numerical calculations are described below.

Step 2: In step two, the data provided in table 1 is aggregated using the T-SFWG operators. The formulation for T-SFOWG and T-SFHG operators are similar to that of T-SFWG operators so we only use T-SFWG operators here. The results obtained are given as follows:

Table 1: (Decision Matrix)

	e_1	e_2	e_3	e_4
P1	(0.6,0.9,0.8)	(0.75,0.4,0.3)	(0.3,0.5,0.7)	(0.4,3,0.7)
P2	(0.8,0.4,0.3)	(0.8,0.4,0.5)	(0.2,0.5,0.9)	(0.3,0.5,0.3)
P3	(0.8,0.4,0.3)	(0.8,0.4,0.4)	(0.6,0.5,0.9)	(0.5,0.5,2)
P4	(0.5,0.2,0.1)	(0.4,0.5,0.9)	(0.5,0.5,0.4)	(0.6,0.2,0.4)

$$T_1 = T - SFWG(T_{11}, T_{12}, T_{13}, T_{14})$$
$$= (0.52267, 0.46685136, 0.7026211)$$
$$T_2 = T - SFWG(T_{21}, T_{22}, T_{23}, T_{24})$$
$$= (0.549495, 0.43734483, 0.6779984)$$
$$T_3 = T - SFWG(T_{31}, T_{32}, T_{33}, T_{34})$$
$$= (0.649254, 0.43734483, 0.6678932)$$
$$T_4 = T - SFWG(T_{41}, T_{42}, T_{43}, T_{44})$$
$$= (0.515169, 0.31622777, 0.6676236)$$

Step 3: Step three is based on evaluating the score value of the aggregated data obtained in step two using score function.

$$S(T_1) = -0.20408$$
$$S(T_2) = -0.14575$$
$$S(T_3) = -0.02425$$
$$S(T_4) = -0.16085$$

Step 4: In step four, the products are ranked based on the score values. The observations are as under.

$$S(T_3) \le S(T_2) \le S(T_4) \le S(T_1)$$

Hence, the product is more likely to be launched according to our observations.

Advantages of Proposed Operations:

The advantage of proposed operations lies in the fact that these operations can handle the data that IFS and PFS could not. For instance, if we look at table 1 it is clear that all the values are purely TSFNs for which means that such numbers cannot be aggregated using the operations of PFSs or any other fuzzy framework. This makes our point clear that the structure of SFS and TSFS are generalizations of IFS and PFS. Another point is that the information aggregated in (Xu, Z. and R.R. Yager, 2006; Cuong, B. C. 2014;) can be aggregated using the aggregation operators of TSFSs so it is better to use the tools of aggregation of SFSs or TSFSs.

CONCLUSION

In this manuscript, the aggregation tools produced in (Mahmood *et al.,* 2018) are improved and their properties of are investigated. The concept of geometric operators is further extended and some OWA and HG aggregation tools are also developed in case where ordered position of arguments or both ordered position and weight of arguments are important. The characteristics like boundedness, idempotency and monotonicity of developed aggregation tools are studied. MADM is explained and the algorithm for MADM problem is studied followed by a numerical problem is discussed in the environment of TSFSs. It is discussed that T-SFSs generalizes IFSs, PFSs and even SFSs proving that it is the most suitable tool to be used in problems involving complex human opinion. In future, the weighted averaging, Einstein and interactive aggregation operators can be developed for TSFSs. Further, some similarity measures for TSFSs can be developed and applied in many real-life problems.

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